On sign-solvable linear systems and their applications in economics

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Abstract

Sign-solvable linear systems are part of a branch of mathematics called qualitative matrix theory. Qualitative matrix theory is a development of matrix theory based on the sign $(-, 0, +)$ of the entries of a matrix. Sign-solvable linear systems are useful in analyzing situations in which quantitative data is unknown or had to measure, but qualitative information is known. These situations arise frequently in a variety of disciplines outside of mathematics, including economics and biology. The applications of sign-solvable linear systems in economics are documented and the development of new examples is formalized mathematically. Additionally, recent mathematical developments about sign-solvable linear systems and their implications to economics are discussed.

1 Introduction

Sign-Solvable Linear Systems are a fundamental part of a branch of mathematics called *qualitative matrix theory*, that is matrix analysis by sign patterns. The development of this topic was motivated by the field of economics.

Many times in economics quantitative data may be hard to obtain or inaccurate. However, qualitative data is usually readily available as a result of theoretically or empirically plausible information. Knowing this information is the primary motivation for qualitative matrix theory, and thus sign-solvable linear systems [5]. An example of an application in economics is qualitative comparative statics, which will be discussed in a subsequent section.

Economics is not the only application of sign-solvable linear systems. Applications occur in the modeling of ecosystems, where again measuring populations is hard leading to inaccurate data but qualitative data is widely known. More on the applications in biology can be found in [2]. There are many other disciplines in which sign-solvable linear systems have applications, most have similar issues with the available data.

2 Qualitative Matrix Theory

2.1 Sign Patterns

The sign of a real number a, denoted by $sgn(a)$, is defined to be $-$ for $a < 0$, 0 for $a = 0$, and $+$ for $a > 0$. The sign pattern of a real matrix A, denoted by $\tilde{A}$ is the $(-, 0, +)$-matrix obtained from A by replacing each entry of A by its sign.

Example 1 *The matrix* $A = \begin{pmatrix} 1 & -2 \\ -3 & -4 \end{pmatrix}$ *has sign pattern* $\tilde{A} = \begin{pmatrix} + & - \\ - & - \end{pmatrix}$.

For any matrix A the *qualitative class* of A, denoted by $Q(A)$, is the set of all matrices whose (i, j)-entry has the same sign as the (i, j)-entry of A.

Example 2 *If* $Q(A) = \{A \in \mathbb{R}^{n\times n} : \tilde{A} = \left(\begin{smallmatrix} + & - \\ - & - \end{smallmatrix}\right)\}$, *then matrix* $A = \left(\begin{smallmatrix} 1 & -2 \\ -3 & -4 \end{smallmatrix}\right) \in Q(A)$.

For a vector b the sign of b and qualitative class of b are similarly defined and denoted by $\tilde{b}$ and $Q(b)$, respectively. Matrix analysis by the sign patterns of matrices and vectors is called qualitative matrix theory.

2.2 Some Signed Properties of Matrices

Other properties of matrices discussed in traditional matrix theory can be formulated in terms of qualitative matrix theory as well. For example the *null space* of a matrix A is the set of all vectors x such that $Ax = 0$. Then we can define the signed null space of A when homogeneous linear system $Ax = 0$ has signed solutions (See Definition 10).

The concept of determinant can also be characterized in terms of sign patterns. We say that a matrix A has *signed determinant* if the determinants of all matrices in $Q(A)$ have the same sign. Recall that the *determinant* of a matrix $A = [a_{ij}]$ of order n is

$$\det A = \sum sgn(\sigma) a_{1i_1} a_{1i_2} \cdots a_{ni_n},$$

where the summation extends over all permutations $\sigma = (i_1, i_2, \ldots, i_n)$ of $1, 2, \ldots, n$ and $sgn(\sigma)$ denotes the sign of the permutation σ (See [2]).

Recall that a square matrix $A \in \mathbb{R}^{n\times n}$ is called *nonsingular* if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$ where I_n is the identity matrix of order n. We state the following well known result with out proof.

Theorem 3 *For $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*

(a) *A is nonsingular;*

(b) $\det A \neq 0$*;*

(c) *The columns of A are linearly independent;*

(d) *The transpose A^T is non-singular; and*

(e) *The equation $Ax = b$ has exactly one solution for each vector $b \in \mathbb{R}^{n\times n}$.*

Nonsingularity can be described in terms of sign patterns as follows:

Definition 4 *A matrix $A \in \mathbb{R}^{n\times n}$ is called sign-nonsingular if A has signed determinant and* $\det A \neq 0$*.*

Definition 5 *([5]) An $m \times n$ matrix with $m \leq n$ is called* totally sign-nonsingular *if the sign of the determinant of each submatrix of order m is determined uniquely by its sign pattern.*

Recall that the rank of a matrix is the maximum number of linearly independent rows (columns). From [8] we have the following theorem:

Theorem 6 *Suppose that the rank of A is equal to the number of rows. Then A has signed null space if and only if it is totally sign-nonsingular.*

A development of qualitative matrix theory is presented by Brualdi and Shader in [2].

2.3 Sign-Solvable Linear System

A linear system $Ax = b$ is called *solvable* if it has at least one solution.

Definition 7 *For a matrix A and a vector b the linear system $Ax = b$ is called sign-solvable if $A'x = b'$ is solvable for each $A' \in Q(A)$ and $b' \in Q(b)$ and its solution is contained in the same qualitative class.*

Recall Cramer's rule as applied to 2×2 matrices. Given $Ax = b$ where A is nonsingular, $x_i = \frac{\det(A_i)}{\det(A)}$ where x_i denotes the ith row of x and A_i is the matrix obtained by replacing the ith column of A with b.

Example 8 *Consider the linear system $Ax = b$ of the form*

$$\begin{pmatrix} -a_1 & -a_2 \\ -a_3 & +a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -b_1 \end{pmatrix} \tag{1}$$

where $a_1, ..., a_4$ and b_1 are positive constants. Note that A has signed determinate and $\det A < 0$*, thus A is nonsingular for any $A' \in Q(A)$. So by Cramer's rule the linear system has the unique solution*

$$x_1 = \frac{\det \begin{pmatrix} 0 & - \\ - & + \end{pmatrix}}{\det \begin{pmatrix} - & - \\ - & + \end{pmatrix}} = \frac{(-)}{(-)} = (+)$$

$$x_2 = \frac{\det \begin{pmatrix} - & 0 \\ - & - \end{pmatrix}}{\det \begin{pmatrix} - & - \\ - & + \end{pmatrix}} = \frac{(-)}{(-)} = (+)$$

for any $A' \in Q(A)$ and $b' \in Q(b)$. Hence the system is sign-solvable.

Definition 9 *For a linear system $Ax = b$ the set of sign patterns of the solutions is denoted by $S(A, b)$. A linear system has signed solutions if for any $A' \in Q(A)$ and $b' \in Q(b)$ the set $S(A', b')$ is the same as $S(A, b)$.*

A linear system that has a unique signed solution is equivalent to the linear system being sign-solvable (See [5]).

A *linear combination of vectors* $x_1, x_2, \ldots, x_n$ is given by $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a_1, \ldots, a_n \in \mathbb{R}$. An *affine combination* of vectors is similarly defined except $\sum a_i = 1$, and a *convex combination* of vectors is also similarly defined except $\sum a_i = 1$ and $a_i \geq 0$ for all i.

Example 10 *Consider the linear system $Ax = b$ of the form*

$$\begin{pmatrix} +a_1 & -a_2 & 0 \\ -a_3 & -a_4 & +a_5 \end{pmatrix} x = \begin{pmatrix} 0 \\ -b_1 \end{pmatrix}$$

where $a_1, ..., a_5 > 0$.

If Ax is viewed as a linear combination of the columns of the matrix A,

$$x_1 \begin{pmatrix} + \\ - \end{pmatrix} + x_2 \begin{pmatrix} - \\ - \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ + \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix}$$

it is clear that the only combinations that guarantee a solution are $x_1 \cdot c_1 + x_2 \cdot c_2 + 0 \cdot c_3 = \left(\begin{smallmatrix} 0 \\ -b_1 \end{smallmatrix}\right)$ and $0 \cdot c_1 + 0 \cdot c_2 + x_3 \cdot c_3 = \left(\begin{smallmatrix} 0 \\ -b_1 \end{smallmatrix}\right)$ where c_1, c_2, c_3 are the columns of A.

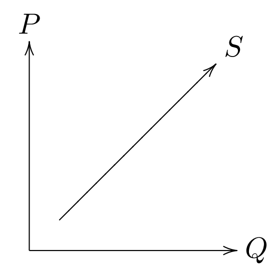
Thus it follows that the solution to the system can be represented by an affine combination of $\left(\begin{smallmatrix} \frac{a_2 b_1}{\alpha} & \frac{a_2 b_1}{\alpha} & 0 \end{smallmatrix}\right)^T$ and $\left(\begin{smallmatrix} 0 & 0 & -\frac{b_1}{a_5} \end{smallmatrix}\right)^T$ where $\alpha = a_1 a_4 + a_2 a_3 > 0$. Hence $S(A, b)$ consists of the five kinds of sign patterns: $\left(\begin{smallmatrix} + & + & 0 \end{smallmatrix}\right)^T$, $\left(\begin{smallmatrix} 0 & 0 & + \end{smallmatrix}\right)^T$, $\left(\begin{smallmatrix} + & + & - \end{smallmatrix}\right)^T$, $\left(\begin{smallmatrix} + & + & + \end{smallmatrix}\right)^T$, and $\left(\begin{smallmatrix} - & - & - \end{smallmatrix}\right)^T$.

3 An Application of Sign-Solvable Linear Systems

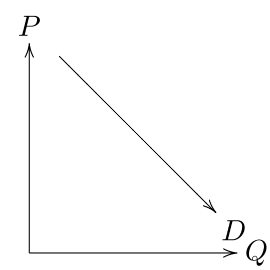
One application of sign-solvable linear systems is qualitative comparative statics. Qualitative comparative statics is the study of the equilibrium of a market before and after a change in either the supply or demand for a good, that is the result of a change in an exogenous parameter. This change is hard to assign a quantitative value, but by theory we know a qualitative value and thus sign-solvability is useful.

3.1 Some Economic Background

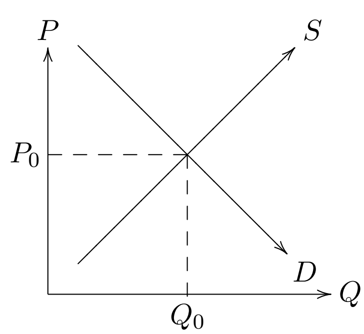
Quantity supplied is the amount of a good or service that a producer is willing and able to supply at a given price. The curve is generally upward sloping, since higher prices would provide motivation for producers to generate more goods. The supply curve is graphically represented by the figure below.



Quantity demanded is the amount of a good or service that a consumer is willing and able to purchase at a given price. In general the demand curve is negatively sloped, because higher prices discourage customers from buying more of a product. Graphically the demand curve is represented by the figure below.



Equilibrium is the price and quantity at which consumers and producers "agree" to exchange goods and services as determined by the market. This value is graphically represented by the intersection of the demand and supply curves as pictured below.

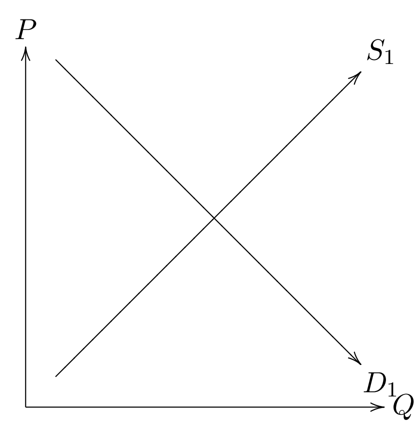


Price and quantity are endogenous variables, that is determined by the economic model. It is important to note that the independent variable is price and the dependent variable is quantity, and that each is graphed on an axis that is opposite of the axis generally used for the independent and dependent variable in mathematics.

Economics is also concerned with exogenous parameters, being forces that can affect the market, but are other than the dependent and independent variable. These exogenous variables are determined outside of a given model. A change in an exogenous parameter causes a shift in demand or supply curve. From the aspect of demand such changes include changes in consumer preferences, prices of related goods, consumer income, an increase in buyers, or an expectation of higher prices.

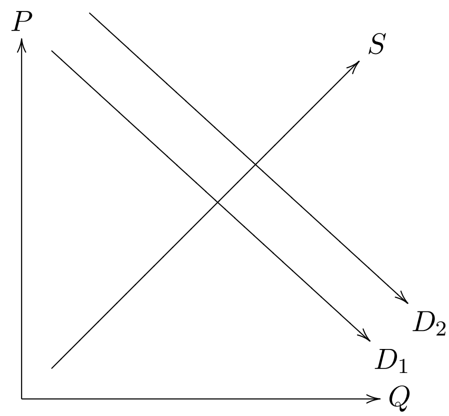
3.2 Simplifying Assumptions

Many times in economics simplifying assumptions are made to better show the results of a specific action, this problem is no different. We will examine a one product market, the supply and demand curves will be represented by straight lines, and be denoted by $S_1(p)$ and $D_1(p)$ respectively.



Let a be a shift parameter that represents a change in consumer preferences. Since consumer preferences are hard to measure, we must rely on theory to explain the value of a. We will assume that if peoples liking for a good increases then demand will increase. That is if a becomes larger than demand increases. When we incorporate this change in demand, the new demand curve is denoted by $D_2(p, a)$.

Then our graph looks like



The goal of qualitative comparative statics is to determine what the change in the equilibrium quantity and price will be. From the graph a good guess can be made, but these curves are strictly linear so the result is not general nor is it a formal proof.

3.3 Determining the Change in Equilibrium

The new equilibrium can be found by solving $S(p) = D(a, p)$. Let x represent the equilibrium quantity found by solving this equation. Then $S(p) - x = 0$ and $D(a, p) - x = 0$. By totally differentiating these two equations with respect to a we get.

$$\frac{\partial S}{\partial p}\frac{dp}{da} - \frac{dx}{da} = 0$$

$$\frac{\partial D}{\partial p}\frac{dp}{da} + \frac{\partial D}{\partial a} - \frac{dx}{da} = 0.$$

Equivalently,

$$\begin{pmatrix} \frac{\partial S}{\partial p} & -1 \\ \frac{\partial D}{\partial p} & -1 \end{pmatrix} \begin{pmatrix} \frac{dp}{da} \\ \frac{dx}{da} \end{pmatrix} = -\begin{pmatrix} 0 \\ \frac{\partial D}{\partial a} \end{pmatrix}. \tag{2}$$

From economic theory we know that if price becomes higher then the quantity supplied increases, thus $S(p)$ is an increasing function with respect p. Also if price becomes higher then the quantity demanded decreases, thus $D(p)$ is a decreasing function with respect to p. By assumption, $D(p, a)$ is increasing with respect to a.

Thus, by the theory and assumptions above we have $\frac{\partial S}{\partial p} > 0$, $\frac{\partial D}{\partial p} < 0$, and $\frac{\partial D}{\partial a} > 0$. Therefore,

$$\begin{pmatrix} \frac{\partial S}{\partial p} & -1 \\ \frac{\partial D}{\partial p} & -1 \end{pmatrix} \begin{pmatrix} \frac{dp}{da} \\ \frac{Dx}{da} \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{\partial D}{\partial a} \end{pmatrix}$$

can be rewritten as

$$\begin{pmatrix} + & -1 \\ - & -1 \end{pmatrix} \begin{pmatrix} \frac{dp}{da} \\ \frac{dx}{da} \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix}$$

Thus by Cramer's rule,

$$\frac{dp}{da} = \frac{det \begin{pmatrix} 0 & -1 \\ - & -1 \end{pmatrix}}{det \begin{pmatrix} + & -1 \\ - & -1 \end{pmatrix}} = \frac{0 - (+)}{(-) - (+)} = \frac{(-)}{(-)} = +$$

and

$$\frac{dx}{da} = \frac{det \begin{pmatrix} + & 0 \\ - & - \end{pmatrix}}{det \begin{pmatrix} + & -1 \\ - & -1 \end{pmatrix}} = \frac{(-) - (0)}{(-) - (+)} = \frac{(-)}{(-)} = +$$

So

$$\begin{pmatrix} \frac{dp}{da} \\ \frac{dx}{da} \end{pmatrix} = \begin{pmatrix} + \\ + \end{pmatrix}.$$

Hence the change in price and quantity is positive with respect to an increase in consumer preferences.

4 General Equilibrium Analysis

The preceding example is of *partial equilibrium analysis*, that is the analysis of the equilibrium of a market with one good, one price, and prices of other goods are considered constant. While this analysis provides some insight into the workings of markets it is not overly realistic of real world markets. In a real world market there are generally many goods, each with a distinct price and the demand (supply) of each good is dependent on the interaction of the various goods and their prices. The development above can be expanded to discuss such markets and is called *general equilibrium analysis*. See [4, 1, 6] for a broader discussion of general equilibrium analysis.

4.1 A Brief Discussion

The goal of general equilibrium analysis is to characterize the economic phenomena occurring in markets. Most interesting to economists are the phenomena related to prices. Prices have a surprising power to convey the information necessary to allocate resources efficiently, that is resources are not wasted nor are they in short supply [1].

Prices are also the only information that is shared among all agents in an economy. Prices along with wealth are the two basic decision-making factors of an agents actions in the market, or *economic behavior*. In other words prices and wealth characterize an individual's demand and supply curves. Summing such information across all agents yields the aggregate supply and demand that lead to the definition of economic equilibrium in the general market [1].

If every good and every price were considered equations for general equilibrium would have the form,

$$
\begin{array}{ccc}
S_1(p_1, p_2, \dots p_n) & \text{and} & D_1(p_1, p_2, \dots p_n) \\
S_2(p_1, p_2, \dots p_n) & \text{and} & D_2(p_1, p_2, \dots p_n) \\
& \vdots & \\
S_n(p_1, p_2, \dots p_n) & \text{and} & D_n(p_1, p_2, \dots p_n)
\end{array}
$$

where $n \in \mathbb{N}$. Again such analysis is not representative of a real world market, since there is no market in which every available good can be exchanged. Generally, the amount of goods in an actual market is considerably smaller than the set of economic goods one can think of; see [1]. For the examples presented here the number of goods will be limited to much smaller numbers, but the insights will still be valuable.

4.2 A Smaller Example

4.2.1 A Three Good Market

Let's consider a three good market described by the following equations,

$$
\begin{array}{llcl}
\text{Good1}: & D_1(p_1, p_2, p_3) & \text{and} & S_1(p_1) \\
\text{Good2}: & D_2(p_1, p_2, p_3) & \text{and} & S_2(p_2) \\
\text{Good3}: & D_3(p_1, p_2, p_3) & \text{and} & S_3(p_3)
\end{array}
$$

Let x_1, x_2, and x_3 be the equilibrium quantities obtained by solving the for the equilibrium in the market corresponding to S_n and D_n for $n \in \{1, 2, 3\}$. Then we have

$$\begin{array}{llll} \text{Good1}: & D_1(p_1, p_2, p_3) - x_1 = 0 & \text{and} & S_1(p_1) - x_1 = 0 \\ \text{Good2}: & D_2(p_1, p_2, p_3) - x_2 = 0 & \text{and} & S_2(p_2) - x_2 = 0 \\ \text{Good3}: & D_3(p_1, p_2, p_3) - x_3 = 0 & \text{and} & S_3(p_3) - x_3 = 0 \end{array}$$

Let's also assume that good 1 and good 3 are *substitutes*, that is when the price of either good increases demand for the other good increases as consumers substitute one good for the other. Additionally let's assume that good 1 and good 2 are *complements*, that is buying one good leads someone to buying the other good. For example hot dogs and hot dog buns are complements. If the price of one complementary good increases then the demand of the other good decreases.

4.2.2 Exploring the Nature of the Market

Let's explore the relationship between good 1 and good 2 by totally differentiating D_1 and S_1 with respect to p_3. We get the following equations:

$$\frac{\partial S_1}{\partial p_1}\frac{dp_1}{dp_3} - \frac{dx_1}{dp_3} = 0$$

$$\frac{\partial D_1}{\partial p_1}\frac{dp_1}{dp_3} + \frac{\partial D_1}{\partial p_2}\frac{dp_2}{dp_3} + \frac{\partial D_1}{\partial p_3} - \frac{dx_1}{dp_3} = 0.$$

and equivalently in matrix form,

$$\begin{pmatrix} \frac{\partial S_1}{\partial p_1} & 0 & -1 \\ \frac{\partial D_1}{\partial p_1} & \frac{\partial D_1}{\partial p_2} & -1 \end{pmatrix} \begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dp_2}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = -\begin{pmatrix} 0 \\ \frac{\partial D_1}{\partial p_3} \end{pmatrix}. \tag{3}$$

Under the assumption that good 1 and good 3 are substitutes we know that $\frac{\partial D_1}{\partial p_3} > 0$ and under the assumption that good 1 and good 2 are complements we know that $\frac{\partial D_1}{\partial p_2} < 0$. So equation (3) can be written qualitatively as,

$$\begin{pmatrix} + & 0 & - \\ - & - & - \end{pmatrix} \begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dp_2}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix} \tag{4}$$

It is easy to see that the linear system in (4) has the signed solution set given below,

$$\begin{pmatrix} - \\ + \\ - \end{pmatrix}, \begin{pmatrix} + \\ 0 \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ + \end{pmatrix}, \begin{pmatrix} + \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 \\ + \\ 0 \end{pmatrix}$$

Clearly this system is not sign-solvable. However, a natural question arising in this context is if any solutions in the set can be eliminated. For example, if the sign of $\frac{dx_1}{dp_3}$ can be determined from some theory outside of the linear system, some of the solutions can be eliminated. In fact the sign of $\frac{dx_1}{dp_3}$ can be determined.

4.2.3 Reducing the Solution Set by Partial Equilibrium Analysis of the Market for Good 1

To eliminate some of the solutions in the solution set of equation (4), we can consider the partial equilibrium analysis of the market for good 1, where $D_1(p_1, p_3) - x_1 = 0$ and $S_1(p_1) - x_1 = 0$. Totally differentiating with respect to p_3 we get:

$$\frac{\partial S_1}{\partial p_1}\frac{dp_1}{dp_3} - \frac{dx_1}{dp_3} = 0$$
$$\frac{\partial D_1}{\partial p_1}\frac{dp_1}{dp_3} + \frac{\partial D_1}{\partial p_3} - \frac{dx_1}{Dp_3} = 0.$$

Equivalently,

$$\begin{pmatrix} \frac{\partial S_1}{\partial p_1} & -1 \\ \frac{\partial D_1}{\partial p_1} & -1 \end{pmatrix} \begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{\partial D_1}{\partial p_3} \end{pmatrix}.$$

Again from economic theory we know that if price of good 1 becomes higher, then the quantity supplied increases, and thus $S_1(p_1)$ is an increasing function with respect p_1. Also if the price of good 1 becomes higher, then the quantity demanded decreases, and thus $D(p_1)$ is a decreasing function with respect to p_1. By assumption of good 1 and good 3 being substitutes, $D(p_1, p_3)$ is increasing with respect to p_3.

Thus, by the theory and assumptions above, we have $\frac{\partial S_1}{\partial p_1} > 0$, $\frac{\partial D_1}{\partial p_1} < 0$, and $\frac{\partial D}{\partial p_3} > 0$. Therefore,

$$\begin{pmatrix} \frac{\partial S_1}{\partial p_1} & -1 \\ \frac{\partial D_1}{\partial p_1} & -1 \end{pmatrix} \begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{Dx_1}{Dp_3} \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{\partial D_1}{\partial p_3} \end{pmatrix}.$$

can be rewritten as

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix} \begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix}.$$

Since this system is qualitatively the same as the system in equation (2), the solution is

$$\begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = \begin{pmatrix} + \\ + \end{pmatrix}.$$

So the solution set from equation (4)

$$\begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dp_2}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = \begin{pmatrix} - \\ + \\ - \end{pmatrix}, \begin{pmatrix} + \\ 0 \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ + \end{pmatrix}, \begin{pmatrix} + \\ + \\ + \end{pmatrix}, \begin{pmatrix} 0 \\ + \\ 0 \end{pmatrix}$$

can be reduced to

$$\begin{pmatrix} \frac{dp_1}{dp_3} \\ \frac{dp_2}{dp_3} \\ \frac{dx_1}{dp_3} \end{pmatrix} = \begin{pmatrix} + \\ 0 \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ + \end{pmatrix}, \begin{pmatrix} + \\ + \\ + \end{pmatrix}$$

since from the partial equilibrium analysis $\frac{dp_1}{dp_3} = +$ and $\frac{dx_1}{dp_3} = +$. In terms of economics, the result above implies that in order for an economics agent to have an effect on the price of good 2, the agent would have to control the magnitude of the change in the price of good 3.

5 Conclusions and Future Research

The problems presented in this paper are very small relative to the size of problems that would arise when examining a larger market. While the problems in this paper were solvable by observations and the implementation of some known results, larger problems that are of interest are significantly harder to solve. In general the computational complexity of such larger problems makes them nearly impossible to solve.

In order to expand the scope of qualitative matrix theory to these larger economic problems new approaches must be developed to deal with the issue of computational complexity. Future research will need to focus on better theory for solving large signed linear systems in order to find solutions to economic problems involving more goods, prices, and other parameters.

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